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On the inverse problem of the calculus of variations in field theory

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Abstract. The inverse problem of the calculus of variations is investigated in the case of field theory. Uniqueness of the action principle is demonstrated for the vector Laplace equation in a non-decomposable Riemannian space, as well as for the harmonic map equation.

1. Introduction

From the standpoint of theoretical physics, the uniqueness aspect of the inverse problem of the calculus of variations is at least as important as its existence aspect. Indeed, it is generally assumed that all 'fundamental' (as opposed to 'phenomenological') physical theories derive from an action principle. What is generally less known, however, is that they may derive in some instances from more than one such principle, and that this leads to different quantum theories. Examples exist in Newtonian mechanics (hydrogen atom—see Henneaux and Shepley (1982)) as well as in field theory (SU(2) chiral model (Nappi 1980)).

It has been argued recently that theories possessing different variational descriptions form a 'set of measure zero' in the space of all theories derivable from a Lagrangian (Henneaux 1982a, b, Henneaux and Shepley 1982). The question arises, however, whether the field theories of physical interest would not precisely belong to that set because of their particular properties (symmetries, locality of the field equations...).

The purpose of this paper is to provide a beginning of an answer to that question. More precisely, given a set of second-order quasi-linear field equations

$$a^{ij}_{AB}(u^C, u^D_k, x^m)u^B_{ij} + b_A(u^C, u^D_k, x^m) = 0 \tag{1.1}$$

belonging to a fairly general class, it is proven that there is one and only one Lagrangian function $\mathscr{L}(u^C, u^D_k, x^m)$ (up to a multiplicative constant corresponding to a change of units and an 'immaterial' divergence) such that the variational equations

$$\delta\mathscr{L}/\delta u^A \equiv -\partial_i(\partial\mathscr{L}/\partial u^A_i) + \partial\mathscr{L}/\partial u^A = 0 \tag{1.2}$$

are equivalent to the system (1.1). x^m ($m = 1, 2, \ldots, n$) are here the independent variables, whereas u^A ($A = 1, \ldots, M$) are the unknown functions. u^A_i and u^A_{ij} are respectively their first- and second-order partial derivatives. Besides, the summation

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convention over repeated indices is adopted and $\partial_i A$ denotes the total partial derivative of the function A with respect to x^i ($\partial_i A = \partial A/\partial x^i + (\partial A/\partial u^B)u_i^B + \ldots$).

The meaning of this result is that the variational description of the usual field theories is essentially unique (when it exists) provided one makes the following restrictions:

(i) the action is the integral $\int \mathrm{d}x^i \mathscr{L}$ of a function $\mathscr{L}$ that only involves x^k, u^A and its first partial derivatives u_i^A (if one gives up that locality requirement, then there is obviously an infinity of variational principles for (1.1), just as there are many functions $R \to R$ which possess the same extrema);

(ii) the variational principle is stated in terms of the variables u^A, which obey second-order partial differential equations (if one introduces additional variables z^α and replaces the system (1.1) by an equivalent first-order system involving u^A and z^α, one may lose uniqueness, as in classical mechanics (see appendix and Havas 1973, Henneaux 1982b, Hojman and Urrutia 1981).

The above locality requirement, which seems natural at first sight, should be somewhat relaxed in a future study, for there exist systems with the following property: they can be described by two sets u^A and $\bar{u}^A$ of independent variables which both obey second-order partial differential equations derivable from a Lagrangian, but which are related through a non-local transformation (see § 6). The corresponding Lagrangians lead to different quantum theories. The present analysis is incapable of deciding which description is the correct one. It is only by additional criteria, concerning for instance they physical meaning of u^A and $\bar{u}^A$, that one can select the 'good' Lagrangian. This problem falls beyond the scope of our study.

The paper is organised as follows. We first consider arbitrary second-order partial differential equations

$$T_A(u^B, u_i^C, u_{jk}^D, x^m) = 0 \tag{1.3}$$

and write down the necessary and sufficient conditions that T_A must obey in order to be the variational derivatives $\delta\mathscr{L}/\delta u^A$ of some function $\mathscr{L}(u^B, u_i^C, u_{jk}^D, x^m)$. When these conditions are fulfilled ($T_A = \delta\mathscr{L}/\delta u^A$), we write explicitly $\mathscr{L}$ in terms of T_A as an expression involving a single line integral (Vainberg 1954, Tonti 1969).

After a general discussion of the inverse problem in the quasi-linear case, we then turn to the Laplace tensor equations in a curved space,

$$\Delta u^A \equiv g^{ij} u^A_{|i|j} = 0 \tag{1.4}$$

where the stroke denotes the covariant derivative in the Riemannian metric g_{ij}. We show that the Lagrangian is essentially unique for scalar field ($u^A \equiv u$), as well as for a vector field ($u^A \equiv u^i$) provided that the metric g_{ij} is non-decomposable.

We then consider a set of scalar fields coupled through a harmonic mapping type of interaction Misner (1978). We again show that the Lagrangian is essentially unique when the metric of the 'image' space is non-decomposable. This case covers the nonlinear σ-model, wich we discuss in some detail in § 6.

Finally, the appendix is devoted to the study of first-order systems.

2. The integrability conditions

Let $T_A(u^B, u_i^C, u_{jk}^D, x^m)$ be n functions of u^B, u_i^C, u_{jk}^D and x^m. The necessary and sufficient conditions for the existence of a functional $S[u^B] = \int \mathscr{L}(u^B, u_i^C, u_{jk}^D, x^m)\, \mathrm{d}^n x$

such that the functional derivatives $\mathrm{D}S/\mathrm{D}u^A(x^m)$ be equal to T_A, i.e. such that

$$\frac{\delta\mathcal{L}}{\delta u^A}\equiv\frac{\partial\mathcal{L}}{\partial u^A}-\partial_i\left(\frac{\partial\mathcal{L}}{\partial u_i^A}\right)+\partial_i\partial_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}^A}\right)=T_A, \tag{2.1}$$

is that the second functional derivatives commute, $\mathrm{D}T_A(x)/\mathrm{D}u^B(x')=\mathrm{D}T_B(x')/\mathrm{D}u^A(x)$. These conditions read explicitly

$$\partial T_A/\partial u_{ij}^B=\partial T_B/\partial u_{ij}^A, \tag{2.2}$$

$$\partial T_A/\partial u_k^B+\partial T_B/\partial u_k^A=2\partial_i(\partial T_B/\partial u_{ik}^A), \tag{2.3}$$

$$\frac{\partial T_A}{\partial u^B}=\frac{\partial T_B}{\partial u^A}-\partial_k\left(\frac{\partial T_B}{\partial u_k^A}\right)+\partial_i\partial_j\left(\frac{\partial T_B}{\partial u_{ij}^A}\right). \tag{2.4}$$

Indeed, when T_A is of the form (2.1), the polynomials

$$\delta T_A\equiv\frac{\partial T_A}{\partial u^B}U^B+\frac{\partial T_A}{\partial u_i^B}U_i^B+\frac{\partial T_A}{\partial u_{ij}^B}U_{ij}^B \tag{2.5}$$

are self-adjoint, in the sense that $V^A T_A(U^B)=U^A T_A(V^B)+\partial_i J^i$ (De Donder 1935, p 204). This implies (2.2)–(2.4). Conversely, when the relations (2.2)–(2.4) hold, the Lagrangian

$$\mathcal{L}=u^A\int_0^1 T_A(\mu u^B,\mu u_i^B,\mu u_{ij}^B,x^m)\,\mathrm{d}\mu \tag{2.6}$$

is such that $\delta\mathcal{L}/\delta u^A=T_A$, as can be checked by direct computation (Tonti 1969). Note that one cannot always get rid of the second-order derivatives u_{ij}^B from $\mathcal{L}$ by partial integration when the number of independent variables exceeds one (example: $\mathcal{L}=u(u_{12}u_{34}-u_{13}u_{24})$ leads to second-order equations which are not quasi-linear).

Upon using (2.3), one can replace (2.4) by

$$\frac{\partial T_A}{\partial u^B}-\frac{1}{2}\partial_k\left(\frac{\partial T_A}{\partial u_k^B}\right)=\frac{\partial T_B}{\partial u^A}-\frac{1}{2}\partial_k\left(\frac{\partial T_B}{\partial u_k^A}\right). \tag{2.7}$$

For one degree of freedom ($M=1$), this equation is an identity.

The inverse problem of the calculus of variations amounts to determining whether one can replace the system (2.1) by an equivalent system $\bar{T}_A(u^B, u_i^C, u_{jk}^D, x^m)=0$ such that $\bar{T}_A$ are the variational derivatives of some function and, if the answer is affirmative, in how many ways this can be done.

In the case of quasi-linear equations,

$$T_A=a_{AB}^{ij}(u^C,u_k^D,x^m)u_{ij}^B+b_A(u^C,u_k^D,x^m), \tag{2.8}$$

one considers only equivalent systems which are also quasi-linear. We will assume from now on that the equations $T_A=0$ are independent for different values of A and that there is no way to obtain relations among u^C, u_k^D, x^m (not involving u_{ij}^B) by combining the T_A's. The conditions for the equivalence of $T_A=0$ with $\bar{T}_A=0$ then read

$$\bar{T}_A=M_A{}^B T_B \tag{2.9}$$

for some non-singular matrix $M_A{}^B(u^C,u_k^D,x^m)$.

For given T_A's, the conditions (2.2)–(2.4) (with T_B replaced by $\bar{T}_B$) become partial differential equations for the 'integrating factors' $M_A{}^B$. It is only in the case $n=1$, M arbitrary (Newtonian mechanics) or $M=1$, n arbitrary (one scalar field) that these equations have been systematically investigated (Newtonian mechanics: Crampin (1981), Henneaux (1982a, b), Marmo and Saletan (1977), Sarlet (1980, 1982) and references therein; one scalar field: Anderson and Duchamp (1982)). For both n and M greater than one, the problem is more complex—a situation not unfamiliar from, for instance, the Hamilton–Jacobi theory (Lepage 1936, Dedecker 1977).

3. Uniqueness of the action principle

Let us now assume that T_A can be written as $\delta\mathscr{L}/\delta u^A$ for some $\mathscr{L}$, i.e. obeys (2.2)–(2.4), and let us investigate under what conditions the action principle is unique. Clearly, the function $\mu\mathscr{L}+\partial_i J^i$ with $\mu \in R_0$ is also an acceptable Lagrangian, but since it leads to the same canonical and hence quantum description, it will be called 'equivalent' to $\mathscr{L}$. We want to determine whether there exist other functions $\bar{\mathscr{L}}$ not of the above form so that the variational equations $\delta\bar{\mathscr{L}}/\delta u^A=0$ are equivalent to $T_A=0$. Such $\bar{\mathscr{L}}$'s will be called inequivalent to $\mathscr{L}$ because they lead to a different canonical structure (although the trajectories are the same).

Theorem. If the only solution to

$$M_A{}^C a_{CB}{}^{ij} = M_B{}^C a_{AC}{}^{ij}, \tag{3.1}$$

$$\frac{\partial M_A{}^C}{\partial u^B_k} T_C + \frac{\partial M_B{}^C}{\partial u^A_k} T_C - 2\partial_i(M_B{}^C) a_{CA}{}^{ik} - M_B{}^C \frac{\partial T_A}{\partial u^C_k} + M_A{}^C \frac{\partial T_C}{\partial u^B_k} = 0, \tag{3.2}$$

$$\begin{aligned}
&\frac{\partial M_A{}^C}{\partial u^B} T_C - \frac{\partial M_B{}^C}{\partial u^A} T_C - \frac{1}{2}\partial_k\left(\frac{\partial M_A{}^C}{\partial u^B_k} T_C\right) + \frac{1}{2}\partial_k\left(\frac{\partial M_B{}^C}{\partial u^A_k} T_C\right) - \frac{1}{2}\partial_k(M_A{}^C)\frac{\partial T_C}{\partial u^B_k} \\
&\quad + \frac{1}{2}\partial_k(M_B{}^C)\frac{\partial T_C}{\partial u^A_k} + M_A{}^C\left[\frac{\partial T_C}{\partial u^B} - \frac{1}{2}\partial_k\left(\frac{\partial T_C}{\partial u^B_k}\right)\right] \\
&\quad - M_B{}^C\left[\frac{\partial T_A}{\partial u^C} - \frac{1}{2}\partial_k\left(\frac{\partial T_A}{\partial u^C_k}\right)\right] = 0
\end{aligned} \tag{3.3}$$

is $M_A{}^C = \mu\delta_A{}^C$, with $\mu \in R_0$ and $\delta_A{}^C$ the Kronecker symbol, then the Lagrangian $\mathscr{L}$ is essentially unique in the sense that all the functions $\bar{\mathscr{L}}$ such that $\delta\bar{\mathscr{L}}/\delta u^A=0 \Leftrightarrow T_A=0$ are equivalent to $\mathscr{L}$.

The demonstration is straightforward: when the T_A's obey (2.2)–(2.4), equations (3.1)–(3.3) are equivalent to (2.2)–(2.4) with $\bar{T}_A = M_A{}^B T_B$ in place of T_A. Thus, a necessary and sufficient condition for $M_A{}^B$ to be an integrating factor is that it fulfils (3.1)–(3.3). But if the only solution to these equations is $\mu\delta_A{}^C$, with $\mu \in R_0$, one must have $\bar{T}_A = \mu T_A$. Accordingly, $\bar{\mathscr{L}}$ is equal to $\mu\mathscr{L}+\partial_i J^i$, by a well known theorem on the calculus of variations (see e.g. De Donder 1935, p 14).

We shall from now on assume that $a_{AB}{}^{ij}$ does not involve the first-order partial derivatives u^D_k.

Theorem. When $\partial a_{AB}{}^{ij}/\partial u^D_k = 0$, the integrating factor $M_A{}^C$ obeys

$$\left(\frac{\partial M_A{}^C}{\partial u^B_k} + \frac{\partial M_B{}^C}{\partial u^A_k}\right) a_{CD}{}^{ij} = \frac{\partial M_B{}^C}{\partial u^D_j} a_{CA}{}^{ik} + \frac{\partial M_B{}^C}{\partial u^D_i} a_{CA}{}^{jk}. \tag{3.4}$$

Indeed, the terms linear in the second-order partial derivatives u^D_{ij} must vanish separately in (3.2). When $\partial a_{AB}{}^{ij}/\partial u^D_k = 0$, these derivatives only occur through the first three terms, which easily leads to the desired result upon taking account of the symmetry property $u^D_{ij} = u^D_{ji}$.

In the cases treated below, as well as in most physical applications, the quantities $a_{AB}{}^{ij}$ possess the factorisable form

$$a_{AB}{}^{ij}(u^D, x^m) = \alpha_{AB}(u^D, x^m) h^{ij}(u^D, x^m) \tag{3.5}$$

where the matrix α_{AB} is non-singular ($\det \alpha_{AB} \neq 0$) so that the equations $T_A = 0$ are independent and all involve effectively the second-order derivatives.

Theorem. If the rank of h^{ij} is greater than or equal to two, the multiplier $M_A{}^B$ is velocity independent ($\partial M_A{}^B/\partial u^C_i = 0$).

Proof. The equations (3.4) are linear in $\partial M_A{}^C/\partial u^B_k$. At a given point x^i, one can, without loss of generality, assume $h^{ij} = \varepsilon^i \delta^{ij}$ (no summation on i), with $(\varepsilon^i)^2 = 1$ or 0, since the equations are invariant under linear transformations in the tangent space and since h^{ij} is symmetric.

Multiplying (3.4) by $h_{ij} \equiv \varepsilon^i \delta_{ij}$ and summing over i, j, one first gets

$$n'(\partial M_{AD}/\partial u^B_k + \partial M_{BD}/\partial u^A_k) = 2\varepsilon^2_k \partial M_{AB}/\partial u^D_k \tag{3.6}$$

(no summation over k). $n' \leqslant n$ is here the number of non-zero eigenvalues of h^{ij}, and the index D in $M_{AD} = M_{DA}$ is lowered with α_{CD}. Eliminating $(\partial M_{AD}/\partial u^B_k) + (\partial M_{BD}/\partial u^A_k)$ from (3.4) by using (3.6), one then easily infers

$$\frac{2\varepsilon^2_k}{n'} \frac{\partial M_{AB}}{\partial u^D_k} h^{ij} = \frac{\partial M_{AB}}{\partial u^D_j} h^{ik} + \frac{\partial M_{AB}}{\partial u^D_i} h^{jk} \tag{3.7}$$

(no summation over k). Multiplying (3.7) by h_{jk} and summing over j and k, one finally finds

$$(2\varepsilon^4_i/n' - \varepsilon^2_i - n')\partial M_{AB}/\partial u^D_i = 0. \tag{3.8}$$

No matter what ε_i is, (3.8) implies

$$\partial M_A{}^B/\partial u^C_i = 0 \tag{3.9}$$

since $n' \geqslant 2$ (and $\det \alpha_{AB} \neq 0$).

If one does not make additional assumptions about T_A, it is difficult to go further in the integration of equations (3.1)–(3.3) (except when $M = 1$ (Anderson and Duchamp 1982)). We will thus treat two particular cases which are wide enough to contain most non-singular systems of physical interest and for which the conditions (3.1)–(3.3) have a clear geometrical meaning. As we shall see, there are instances where non-trivial integrability factors exist which involve u^A or x^i.

The first example to be considered is the tensor Laplace equation,

$$T_A = \sqrt{g}\,\alpha_{AB} u^B_{|i|j} g^{ij} = \sqrt{g}\,\alpha_{AB} \Delta u^B \tag{3.10}$$

where the stroke denotes the covariant derivative in the Riemannian metric g_{ij} (of determinant $g \neq 0$; $h^{ij} = \sqrt{g} g^{ij}$). u^B is a tensor field and α_{AB} the 'tensor product' metric induced by g_{ij} in the linear space of the u^B's (α_{AB} is a combination of g_{ij} and only involves x^m—see below). $u^B{}_{|i}$ reads explicitly

$$u^B{}_{|i} = u^B{}_i + \Gamma^B{}_{Ci} u^C \tag{3.11}$$

where $\Gamma^B{}_{Ci}$ are combinations of the Christoffel symbols.

The second case to be treated is the 'harmonic map' case,

$$T_A = \sqrt{g}\,\alpha_{AB}(\Delta u^B + \Gamma^B{}_{CD} u_i^C u_j^D g^{ij}) \tag{3.12}$$

where u^B are now M scalar fields which define a mapping from the spacetime manifold (x^i) into the curved M-dimensional space (u^B) endowed with the line element

$$\mathrm{d}\sigma^2 = \alpha_{AB}(u^C)\,\mathrm{d}u^A\,\mathrm{d}u^B \tag{3.13}$$

(Misner 1978). $\Gamma^B{}_{CD}(u^E)$ are the corresponding Christoffel symbols.

The first example is linear in the u^A's. The second is not.

4. The tensor Laplace equation

With $a_{AB}{}^{ij}$ given by (3.5), the condition (3.2) reads

$$\left(-2\frac{\partial M_B{}^C}{\partial u^D} u_i^D - 2\frac{\partial M_B{}^C}{\partial x^i}\right)\alpha_{CA} g^{ik}\sqrt{g} - M_B{}^C \frac{\partial T_A}{\partial u_k^C} + M_A{}^C \frac{\partial T_C}{\partial u_k^B} = 0 \tag{4.1}$$

since $\partial M_A{}^C/\partial u_k^D = 0$. Because T_A is here linear in u_k^C, the only term which contains the derivatives u_i^D is the first one, and it must vanish. Hence one finds that $M_A{}^B$ is independent of u^C,

$$\partial M_A{}^B/\partial u^C = 0. \tag{4.2}$$

Using this result, one can rewrite (4.1) as

$$M_{AB|i} = 0 \tag{4.3}$$

where we have taken the symmetry conditions (4.1) into account, as well as the fact that the covariant derivatives of α_{AB} vanish.

When (4.3) holds, the remaining condition (3.3) is satisfied: the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\sqrt{g}\, M_{AB} u_{|i}^A u_{|j}^A g^{ij} \tag{4.4}$$

yields the correct equations of motion $M_A{}^B T_B = 0$ ($\det M_{AB} \neq 0$). Thus, the complete resolution of the inverse problem amounts to study the equation (4.3).

(*a*) *Scalar case* ($u^A \equiv u$, $M = 1$, $\alpha_{AB} = 1$)

The condition (4.3) becomes $M_i = 0$, i.e. $M =$ constant. The Lagrangian is essentially unique in agreement with a theorem by Anderson and Duchamp (1982).

(*b*) *Vector case* ($u^A \equiv u^i$, $M = n$, $\alpha_{AB} = g_{ij}$)

The condition (4.3) reads

$$M_{ij|k} = 0. \tag{4.5}$$

In a Riemannian space (positive definite metric), a non-trivial symmetric tensor obeying (4.5) exists if and only if the metric is decomposable, i.e. if there exists a coordinate system t^a, $y^{a'}$ ($a = 1, \ldots, \bar{n} < n$; $a = \bar{n}+1, \ldots, n$) such that

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = g_{ab}(t^c)\,\mathrm{d}t^a\,\mathrm{d}t^b + g_{a'b'}(y^{c'})\,\mathrm{d}y^{a'}\,\mathrm{d}y^{b'}. \tag{4.6}$$

Going to an orthonormal frame in which M_{ij} is diagonal, one can indeed define a new tensor $\bar{M}_{ij}$ by rescaling the eigenvalues of M_{ij} to 0 or 1 in such a way that (i) $\bar{M}_{ij|k}=0$, (ii) $\bar{M}_{ik}\bar{M}^k{}_j=\bar{M}_{ij}$ (idempotency) and (iii) $\bar{M}_{ij}=\bar{M}_{ji}$ (with $\bar{M}_{ij}\neq\mu g_{ij}$). Equation (4.6) then follows (Petrov 1969, p 350). The reducibility of the metric is a very strong condition. It is not fulfilled by a generic metric, for which the only solution to (4.5) is μg_{ij}, $\mu\in R$. In the pseudo-Riemannian case, the condition (4.5) is also very strong (although it is not equivalent to (4.6) when some of the eigenvectors of M_{ij} are null). One can thus conclude that the uniqueness of the Lagrangian—up to the above equivalence relation—is generally guaranteed for the vector Laplace equation.

In the particular case of flat space, which is obviously decomposable, the general solution to (4.5) is $M_{ij}=$constant (in Minkowskian coordinates). There are thus $n(n+1)/2$ inequivalent families of Lagrangians.

(*c*) *Second-order tensor case* ($u^A\equiv u^{ij}$, $M=n^2$)
The equation (4.3) reads this time

$$M_{ijkl|m}=0 \tag{4.7}$$

with $M_{ijkl}=M_{klij}$. In the generic case, its general solution depends on three parameters,

$$M_{ijkl}=\alpha(g_{ik}g_{jl}+g_{il}g_{jk})+\beta(g_{ik}g_{jl}-g_{il}g_{jk})+\gamma g_{ij}g_{kl} \tag{4.8}$$

(except in two and four dimensions, where there is a fourth parameter corresponding to $\varepsilon_{ij}g_{kl}+\varepsilon_{kl}g_{ij}$ and ε_{ijkl}, respectively). The Lagrangian (4.4) is thus not unique. One recovers uniqueness, however, by requiring u^{ij} to be symmetric and traceless, or antisymmetric (and derivable from a potential in four dimensions).

A similar degeneracy occurs for the higher-order tensor Laplace equations (which are physically less interesting).

Let us finally note that the condition $M_{AB|i}=0$, which expresses that M_{AB} is covariantly conserved, should be compared with a theorem by Hojman and Harleston (1981), which relates integrating factors with constants of the motion in classical mechanics (see also Cariñena and Ibort 1983, Crampin 1983, Gonzalez-Gascón 1982, Henneaux 1981, Lutzky 1982 etc).

5. The harmonic map equation

Inserting (3.12) into (4.1), one finds

$$-2(\partial M_B{}^C/\partial u^D+M_B{}^F\Gamma^C{}_{FD}-M_F{}^C\Gamma^F{}_{BD})u_i^D\alpha_{CA}g^{ik}\sqrt{g}-2(\partial M_B{}^C/\partial x^i)\alpha_{CB}g^{ik}\sqrt{g}=0 \tag{5.1}$$

where again the symmetry of M_{AB} has been used. The terms linear in u_i^D and independent of u_i^D must vanish separately, which implies

$$\partial M_B{}^C/\partial x^i=0, \qquad M_B{}^C{}_{;D}=0, \tag{5.2}$$

where ; is the covariant derivative in the metric α_{AB}. The equations (5.2) are the necessary and sufficient conditions that $M_A{}^B$ must obey in order to be an integrating factor (with det $M_A{}^B\neq 0$). The Lagrangians read

$$\mathscr{L}=\sqrt{g}\,g^{ij}M_{AB}u_i^Au_j^B. \tag{5.3}$$

If the Riemannian metric of the 'image' space is non-decomposable, as it is generally the case, the only solution to (5.2) is $M_B{}^C=\mu\delta_B{}^C$ and the Lagrangian (5.3) with $M_{AB}=\alpha_{AB}$ is again essentially unique.

6. The nonlinear σ-model

The nonlinear σ-model being a particular 'harmonic map system' (the image manifold is O(3)), it follows from § 5 that its Lagrangian is essentially unique. This means that there is one and only one equivalence class of Lagrangians which yield local equations (in u^A) equivalent to the usual equations of the nonlinear σ-model.

Now, let V be the space of (regular) functions v of R^2 obeying $v(0,0)=0$, $\Delta v(x^1, 0)=0$ and let W be the space of (regular) functions w of R^2 obeying $w(0,0)=0$, $\Delta w(0, x^2)=0$. The correspondence implicitly defined by

$$\begin{cases} w_1 = v_2, \\ w_2 = -v_1 + \displaystyle\int_0^{x^1} \Delta v(z, x^2)\,\mathrm{d}z, \end{cases} \qquad \begin{cases} v_2 = w_1, \\ v_1 = -w_2 + \displaystyle\int_0^{x_2} \Delta w(x^1, z)\,\mathrm{d}z, \end{cases} \tag{6.1}$$

is a non-local bijective correspondence between the spaces V and W. For instance, $(x^1)^2-(x^2)^2$ is mapped on $-2x^1x^2$ whereas $x^1+\frac{1}{3}(x^2)^3$ is mapped on $\frac{1}{2}x^1(x^2)^2-x^2$.

It is easy to check that the subspace of V obeying $\Delta v=0$ is mapped on the subspace of W defined by $\Delta w=0$ and *vice versa*. Thus, although the correspondence (6.1) is non-local, if v obeys the local (in v) Laplace equation $\Delta v=0$, w also obeys the local (in w) Laplace equation $\Delta w=0$. But the action integral $\frac{1}{2}\int (\nabla v)^2\,\mathrm{d}^2x$ which leads to the Laplace equation for v is clearly inequivalent to $\frac{1}{2}\int (\nabla w)^2\,\mathrm{d}^2x$. For instance, the Hamiltonian structures which they yield are different: although the 'equal time' bracket $[v_1(x^1, x^2), v_2(x^1, \bar{x}^2)]$ is well defined and equal to $\delta_2(x^2, \bar{x}^2)$ with the first Lagrangian, it is undefined if one adopts the second one (one would have to go 'on shell'; x^1 is referred to as the 'time', v_1 being the 'velocity'). The criterion of locality is accordingly too weak to select one single Lagrangian in such circumstances: one needs to know what are the 'basic' variables.

Nappi (1980) has found a generalisation of the above transformation in the case of the two-dimensional σ-model. Thus, one can generate for that model two inequivalent Lagrangians, each local in its corresponding variables (and each unique if one insists on locality in terms of its set of basic variables), but leading to different quantisations.

It is not known whether such transformations exist for all dynamical models. We feel, however, that additional physical criteria (e.g. switching on of an appropriate interaction) can decide which Lagrangian to adopt when there remains some ambiguity. This problem falls, however, beyond the scope of this paper.

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Appendix

We study here first-order partial differential equations

$$T_A(u^B, u_i^C, x^m) = 0 \tag{A1}$$

derivable from a Lagrangian.

It follows from § 2 that T_A is a variational derivative if and only if

$$\partial T_A/\partial u^B_k + \partial T_B/\partial u^A_k = 0, \tag{A2}$$

$$\partial T_A/\partial u^B = \partial T_B/\partial u^A - \partial_k(\partial T_B/\partial u^A_k). \tag{A3}$$

When spelled out in detail, (A3) reads

$$\frac{\partial T_A}{\partial u^B} = \frac{\partial T_B}{\partial u^A} - \frac{\partial^2 T_B}{\partial u^C_m\, \partial u^A_k} u^C_{km} - \frac{\partial^2 T_B}{\partial u^C\, \partial u^A_k} u^C_k - \frac{\partial^2 T_B}{\partial x^k\, \partial u^A_k}. \tag{A4}$$

The equality (A4) can hold only if the term containing the second-order derivatives vanishes, i.e.

$$\partial^2 T_B/\partial u^C_m\, \partial u^A_k + \partial^2 T_B/\partial u^C_k\, \partial u^A_m = 0. \tag{A5}$$

These equations have already been encountered in a related context (Dedecker 1978, Henneaux 1978 appendix A, Hojman 1983), and it is easy to see that their general solution is

$$T_A = \sum_{\alpha=0}^{\Gamma} \frac{1}{\alpha!} \overset{(\alpha)}{a}{}^{i_1 \ldots i_\alpha}_{AC_1 \ldots C_\alpha} u^{C_1}_{i_1} u^{C_2}_{i_2} \ldots u^{C_\alpha}_{i_\alpha} \tag{A6}$$

where the coefficients $\overset{(\alpha)}{a}{}^{i_1 \ldots i_\alpha}_{AC_1 \ldots C_\alpha}$ only involve u^D and x^k and are odd for the permutations of the indices $C_1, \ldots, C_\alpha$ ($i_1, \ldots, i_\alpha$ being fixed) as well as for the permutations of the indices $i_1, \ldots, i_\alpha$ ($C_1, \ldots, C_\alpha$ being fixed). Γ is equal to n when $n \leq M$ or to M when $M \leq n$ ($\Gamma = \min(n, M)$).

Because of (A2), $\overset{(\alpha)}{a}{}^{i_1 \ldots i_\alpha}_{AC_1 \ldots C_\alpha}$ must also be antisymmetric for the permutation of the index A with any of the C's, so that these functions can be viewed as the components of the $(\alpha+1)$-forms

$$\overset{(\alpha)}{a}{}^{i_1 \ldots i_\alpha} = [1/(\alpha+1)!] \overset{(\alpha)}{a}{}^{i_1 \ldots 1_\alpha}_{C_1 C_2 \ldots C_{\alpha+1}}\, \mathrm{d}u^{C_1} \wedge \mathrm{d}u^{C_2} \wedge \ldots \wedge \mathrm{d}u^{C_{\alpha+1}} \tag{A7}$$

($\binom{\alpha}{n}$ such forms for a given α).

The geometric meaning of the remaining equations (A4) can now be written more explicitly. One gets

$$\mathrm{d}\overset{(\alpha)}{a}{}^{i_1 \ldots i_\alpha} + \partial \overset{(\alpha+1)}{a}{}^{ki_1 \ldots i_\alpha}/\partial x^k = 0 \tag{A8}$$

with the convention $\overset{(\Gamma+1)}{a} = 0$.

When the equations (A6)–(A8) hold, the Lagrangian is given by

$$L = \sum_{\alpha=0}^{\Gamma} \frac{1}{\alpha!} \overset{(\alpha)}{b}{}^{i_1 \ldots i_\alpha}_{C_1 \ldots C_\alpha} u^{C_1}_{i_1} \ldots u^{C_\alpha}_{i_\alpha} \tag{A9}$$

where the α-forms $\overset{(\alpha)}{b}{}^{i_1 \ldots i_\alpha}$

$$\overset{(\alpha)}{b}{}^{i_1 \ldots i_\alpha} = (1/\alpha!) b^{i_1 \ldots i_\alpha}_{C_1 \ldots C_\alpha}\, \mathrm{d}u^{C_1} \wedge \ldots \wedge \mathrm{d}u^{C_\alpha} \tag{A10}$$

are antisymmetric in the indices $i_1 \ldots i_\alpha$ and are determined by

$$\mathrm{d}\overset{(\alpha)}{b}{}^{i_1 \ldots i_\alpha} = \overset{(\alpha)}{a}{}^{i_1 \ldots i_\alpha} + \partial \overset{(\alpha+1)}{b}{}^{ki_1 \ldots i_\alpha}/\partial x^k \tag{A11}$$

(with $\overset{(\Gamma+1)}{b} = 0$).

In the case of quasi-linear equations,

$$T_A = \overset{(1)}{\theta}{}^i{}_{AB}(u^C, x^m)u^B_i + \overset{(0)}{\theta}{}_A(u^C, x^m), \tag{A12}$$

a necessary and sufficient condition for the existence of a variational principle leading to quasi-linear equations equivalent to $T_A = 0$ is, as we have just seen, that the equations in $M_A{}^B(u^C, x^m)$

$$M_A{}^C \overset{(1)}{\theta}{}^i{}_{CB} = -M_B{}^C \overset{(1)}{\theta}{}^i{}_{CA}, \tag{A13}$$

$$\mathrm{d}(M\overset{(1)}{\theta}{}^i) = 0, \tag{A14}$$

$$\mathrm{d}(M\overset{(0)}{\theta}) + \partial_k(M\overset{(1)}{\theta}{}^k) = 0, \tag{A15}$$

possess a solution with non-vanishing determinant. Here, $M\overset{(1)}{\theta}{}^i$ and $M\overset{(0)}{\theta}$ are the forms $\frac{1}{2}M_A{}^C \overset{(1)}{\theta}{}^i{}_{CB}\,\mathrm{d}u^A \wedge \mathrm{d}u^B$ and $M_A{}^C \overset{(0)}{\theta}{}_C\,\mathrm{d}u^A$, respectively.

It is not our purpose to give here a full discussion of equations (A13)–(A15). We will simply integrate them explicitly in the case of the two-dimensional scalar Laplace equation $\Delta u = 0$ rewritten in first-order form,

$$T_0 \equiv -u^1_1 - u^2_2 = 0, \tag{A16}$$

$$T_1 \equiv u^0_1 - u^1 = 0, \tag{A17}$$

$$T_2 \equiv u^0_2 - u^2 = 0, \tag{A18}$$

$(u \equiv u^0)$. These equations derive from

$$L = u^1 u^0_1 + u^2 u^0_2 - \tfrac{1}{2}[(u^1)^2 + (u^2)^2]. \tag{A19}$$

One has

$$\overset{(1)}{\theta}{}^1{}_{AB} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \overset{(1)}{\theta}{}^2{}_{AB} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A20}$$

$$\overset{(0)}{\theta}{}_A = \alpha(0 \quad -u^1 \quad -u^2), \tag{A21}$$

where α is a real parameter equal to one in our case, but which will be allowed different values later on.

The first equation (A13) imposes that M must be a multiple of the identity,

$$M = \lambda I \tag{A22}$$

with $\lambda = \lambda(u^C, x^m)$. Equation (A15) then reads

$$\mathrm{d}\lambda \wedge \overset{(0)}{\theta} + (\partial_k \lambda)\overset{(1)}{\theta}{}^k = 0 \tag{A23}$$

i.e.

$$\partial\lambda/\partial x^1 + \alpha u^1\, \partial\lambda/\partial u^0 = 0, \tag{A24}$$

$$\partial\lambda/\partial x^2 + \alpha u^2\, \partial\lambda/\partial u^0 = 0, \tag{A25}$$

$$\alpha(u^1\, \partial\lambda/\partial u^2 - u^2\, \partial\lambda/\partial u^1) = 0. \tag{A26}$$

Taking the Lie brackets of these expressions, one finds the additional equation

$$\partial\lambda/\partial u^0 = 0 \qquad (\alpha \neq 0). \tag{A27}$$

Hence, λ is a function of $R \equiv (u^1)^2 + (u^2)^2$ only when $\alpha \neq 0$.

Requiring finally that $\mathrm{d}(\lambda \overset{(1)}{\theta}{}^i)$ vanishes too, one finds that λ is a constant ($\alpha \neq 0$). The above Lagrangian is thus essentially unique. When $\alpha = 0$, λ can be a function of u^0 and the Lagrangians are given by $\lambda(u^0)[u^1 u^0_1 + u^2 u^0_2]$. This situation should be compared with Newtonian mechanics ($n = 1$), where equations (A13)–(A15) always possess an infinity of inequivalent solutions.

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